

On the Additive Volume of Sets of Integers

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ABSTRACT:

The study of the additive volume of sets can be reduced to the case of one-dimensional sets. The exact values of the volume of extremal sets are given as a conjecture.

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The exposition in this paper is neither complete, nor perfected. Nevertheless, I decided to publish it because of the general interest and the importance of the theme and the new ways I propose to treat it.

1 Introduction

Our main object of study is a finite set $A \subset \mathbb{Z}$. We say that A is in normal form when

$$A = \{a_0 = 0 < a_1 < \cdots < a_{k-1}\}$$

and $\text{g.c.d.}(a_1, \dots, a_{k-1}) = 1$.

We call *doubling* of A the set

$$A + A := 2A = \{x : x = a + b, a, b \in A\}.$$

Our exposition will use two basic numbers characterising the set A : the cardinality of A

$$k := |A|;$$

the cardinality of the doubling of A

$$T := |2A|.$$

The main notions we will use are those of Freiman homomorphism and isomorphism, dimension, parallelepiped, n -dimensional arithmetic progression, and the additive volume of the set A .

The main previous result relevant to us is Freiman's Theorem, see Nathanson [13].

The purpose of this paper is to show how the study of sets of arbitrary dimension can be carried out most effectively by transition to the case of one-dimensional sets.

For the convenience of readers who are not familiar with the above notions we will give now the main definitions.

Definition 1 (Nathanson [13], p. 233) Let G and H be Abelian groups and let $A \subseteq G$ and $B \subseteq H$. A map $\phi : A \rightarrow B$ is called a *Freiman homomorphism* of order 2 if

$$\phi(a_1) + \phi(a_2) = \phi(a'_1) + \phi(a'_2) \tag{1.1}$$

for all $a_1, a_2, a'_1, a'_2 \in A$ such that

$$a_1 + a_2 = a'_1 + a'_2. \tag{1.2}$$

In this case the induced map $\phi^{(2)} : 2A \rightarrow 2B$, given by

$$\phi^{(2)}(a_1 + a_2) = \phi(a_1) + \phi(a_2), \tag{1.3}$$

is well defined.

If $\phi : A \rightarrow B$ is a one-to-one correspondence such that (3) holds if and only if (2) holds, then ϕ is a *Freiman isomorphism* of order 2 and the induced map $\phi^{(2)} : 2A \rightarrow 2B$ is also one-to-one.

In this paper we will simply call these notions homomorphism and isomorphism.

Definition 2. A *parallelepiped* is a set

$$D = (a_1, \dots, a_n) + \{(x_1, \dots, x_n) \in \mathbb{Z}^n : 0 \leq x_i < h_i, i = 1, \dots, n\},$$

where $h_i \in \mathbb{Z}$, $i = 1, \dots, n$.

I. Ruzsa introduced the useful notion of n -dimensional arithmetic progression.

Definition 3. Let a, q_1, \dots, q_n be elements of \mathbb{Z} , and let l_1, \dots, l_n be positive integers. The set

$$Q = Q(a, q_1, \dots, q_n; l_1, \dots, l_n) = \{a + x_1 q_1 + \dots + x_n q_n : 0 \leq x_i < l_i \text{ for } i = 1, \dots, n\}$$

is called an *n -dimensional arithmetic progression* in the group \mathbb{Z} . We usually take $a = (a_1, \dots, a_n) = 0$.

The length of Q is defined to be $l(Q) = l_1 \cdots l_n$. Clearly, if we denote the cardinality of a set X by $|X|$, then

$$|Q| \leq l(Q).$$

It is clear that an n -dimensional arithmetic progression is the image of a suitable n -dimensional parallelepiped under a homomorphism. If the latter is an isomorphism, the progression is said to be *proper*.

We quote Melvyn B. Nathanson (see [13], p. 231).

“Freiman’s theorem asserts that if A is a finite set of integers such that the sumset $2A$ is small, then A is a large subset of a multidimensional arithmetic progression”.

More precisely, the following results holds.

Theorem 1. (Freiman) *Let A be a finite set of integers such that*

$$|2A| \leq c|A|. \tag{1.4}$$

Then there exists integers $a, q_1, \dots, q_n; l_1, \dots, l_n$ such that, if Q is as in Definition 3, then $A \subseteq Q$, where

$$|Q| \leq c'|A| \tag{1.5}$$

and n and c' depend only on c .

After this theorem, many results were obtained, gradually improving the estimate of the constant $c' = c'(c)$. For instance:

Y. Bilu [1]:

$$c' \leq (2d)^{\exp \exp \exp(9c \log(2c))}.$$

I. Rusza [14]:

$$c' \leq \exp\{c^{c^c}\}$$

Mei Chu-Chang [2]:

$$c' \leq \exp\{Kc^2(\log c)^3\},$$

where K is a constant.

T. Sanders [15]:

$$c' \leq \exp\{O(c^{7/4} \log^3 c)\}.$$

S. V. Konyagin [11]:

$$c' \leq \exp\{c \log c\}.$$

T. Schoen [16]:

$$c' \leq \exp\{c^{1+D(\log c)^{-1/2}}\},$$

where D is a positive constant.

The notion of isomorphism enables us now to introduce the notion of dimension of a set A .

Definition 4. We say that a set $A \subset \mathbb{Z}^n$ has *dimension* d if there exists a set $B \subset \mathbb{Z}^d$ isomorphic to A such that B is not contained in any hyperplane of \mathbb{Z}^d , and d is the maximal number with this property. We denote it by $\dim A$.

We are now ready to introduce the notion of additive volume of a set A with given $|2A|$.

Definition 5. Let A have dimension d . Among the sets B as in the preceding definition, take a B such that the number $V(B)$ of integer points in the convex hull of B is minimal. We call this number $V(B)$ the *additive volume* of the set A , and denote it by $V(A)$, observing immediately that it is simultaneously the volume of any other set isomorphic to A .

Let us now formulate the main idea of the present paper. Two sets A and A_1 with the same characteristics k and T , i.e., such that $k = |A| = |A_1|$ and $T = |2A| = |2A_1|$ are not at all necessarily isomorphic, and their volumes $V(A)$ and $V(A_1)$ do not have to be equal. One can ask if, generally, is it possible, instead of a given family of isomorphic sets, to consider another family, with the same or close characteristic values k and T , and which is simpler to study? One of our main results reads:

Theorem 2. *Let $\dim A = 2$. Then there exist a set $A_1 \subset \mathbb{Z}$ and a one-to-one homomorphism $\phi : A \rightarrow A_1$ such that*

$$\dim A_1 = 1,$$

$$T(A_1) < T(A),$$

and

$$V(A_1) > V(A).$$

2 From dimension 2 to dimension 1

2.1. When we compute $T = |2A|$ we have to take into account relations of the form

$$a_i + a_j = 2a_s \tag{2.1}$$

and

$$a_i + a_j = a_r + a_s \tag{2.2}$$

among the elements of $2A$.

S. Konyagin and V. Lev [12] found a very useful representation of relations (2.1) and (2.2) by means of k -dimensional integer vectors. Let us explain it on the following example, which makes clear the general case. Take $k = 7$ and consider the relations

$$a_0 + a_2 = 2a_1 \tag{2.3}$$

and

$$a_1 + a_5 = a_2 + a_4. \tag{2.4}$$

We associate to (2.3) and (2.4) the 7-dimensional vectors

$$(1, -2, 1, 0, 0, 0, 0)$$

and

$$(0, 1, -1, 0, -1, 1, 0),$$

respectively.

In the general case a system of relations (2.1) and (2.2) similarly leads to a collection of k -dimensional integer vectors. We can now look for a maximal linearly independent system of vectors in this collection, to which corresponds a maximal linearly independent system of relations; the cardinality of this system of vectors (or relations) is denoted by $\lambda(A)$.

Theorem 3. (Konyagin and Lev) *For any set $A \subseteq \mathbb{Z}^n$ of k elements, the dimension of A is equal to*

$$\dim A = k - 1 - \lambda(A).$$

We can use the above connection and tools of linear algebra in order to compute the number $\lambda(A)$ and find explicitly a maximal system of linearly independent relations.

2.2. Let A be a set for which there is $D(A)$ such that $\bar{a} = 0$ and $h_2 = 2$, i.e., A lies in the union of the two lines $y = 0$ and $y = 1$, and $0 \leq x \leq h_1 - 1$.

Which points in $D(A)$ are in A ?

Let $A' = \{\bar{z} := (x, y) : y = 0\}$ and $A'' = \{\bar{z} = (x, y) : y = 1\}$. The sets A' and A'' can be arbitrarily moved along the axis x . So, let us assume that $(0, 0)$ and $(0, 1)$ lie in A . Suppose also that $(x_0, 0)$ and $(x_1, 1)$ are in $D(A)$, and x_0 and x_1 are maximal: they are the right end-points of the segments that contain A . One can consider that $x_0 \geq x_1$. Therefore, $(h_1 - 1, 0) \in A$.

$$\begin{array}{ccc} (0, 1) & & \\ \bullet & & \\ | & \text{-----} & \bullet \\ (0, 0) & & (h_1 - 1, 0) \end{array} \quad (2.5)$$

Thus, summarizing, the points $(0, 0)$, $(0, 1)$, and $(h_1 - 1, 0)$ lie in A , and the point $(b, 1)$ with $b \leq h_1 - 1$ is the extremal right point on the line $y = 1$. It can even coincide with $(0, 1)$,

Let us project A'' on the line $y = 0$ parallel to the vector $\ell = (\ell_1, -1)$, where first $\ell_1 > 2h_1 - 2$, and denote the projection map by ϕ :

$$\begin{array}{ccccccc} \bullet & \text{-----} & \bullet & \text{-----} & \bullet & \text{-----} & \bullet \\ (0, 0) & & (h_1 - 1, 0) & & (\ell_1, 0) & & (\ell_1 + b, 0) \end{array} \quad (2.6)$$

Then A_{proj} is isomorphic to A .

Now let us put

$$\ell_1 = 2h_1 - 2.$$

The matrix of the map ϕ is

$$M = \begin{pmatrix} 1 & 0 \\ 2(h_1 - 1) & 0 \end{pmatrix},$$

so that

$$(x, y) \begin{pmatrix} 1 & 0 \\ 2(h_1 - 1) & 0 \end{pmatrix} = (x + 2(h_1 - 1)y, 0);$$

note that

$$\phi(x, 1) = (x + 2(h_1 - 1), 0).$$

We have $A_1 := A_{\text{proj}} = A' \cup A''_{\text{proj}}$, where

$$A' \subset [0, h_1 - 1],$$

$$A''_{\text{proj}} \subset [2h_1 - 2, 2h_1 - 2 + b].$$

Note that $\text{proj}(0, 1) = (2h_1 - 2, 0)$ and that in A_{proj} we have a relation, namely

$$(0, 0) + (2h_1 - 2, 0) = 2(h_1 - 1, 0), \quad (2.7)$$

which we did not have for the corresponding points of A :

$$(0, 0) + (0, 1) \neq 2(h_1 - 1, 0).$$

Therefore, we have

$$V(A) = h_1 + b + 1$$

and

$$V(A_1) = 2h_1 - 1 + b,$$

so that

$$V(A) < V(A_1).$$

Moreover,

$$T(A_1) = T(A) - 1.$$

In our case we have $\dim A = 2$: we added one new relation, (2.7), and so according to Theorem 3, $\dim A_1 = 1$. Thus in our trivial case Theorem 2 is proved.

2.3. Let us give simple examples that illustrate the main idea of the proof of Theorem 2 in the case when $3 \leq h_2 \leq h_1$.

Example 1. Take the following set A :

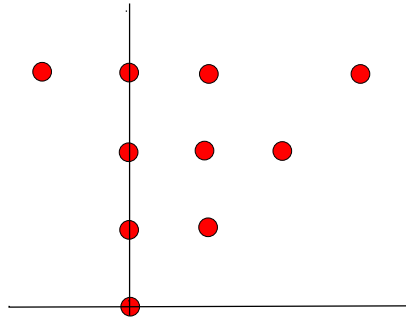


Figure 1

$$A = \{(-1, 3), (0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}.$$

Here we have $k = 10$ and $T = 32$.

Let us project the set A onto the line $x = 0$ parallel to the vector $(1, 6)$. Then we have $\phi(A) = A_1$, where $\phi(-1, 3) = (0, 9)$, $\phi(1, 1) = (0, -5)$, $\phi(1, 2) = (0, -4)$, $\phi(1, 3) = (0, -3)$, $\phi(2, 2) = (0, -10)$, $\phi(3, 3) = (0, -15)$.

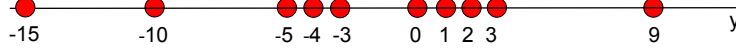


Figure 2

The map ϕ is a one-to-one homomorphism. ϕ is *not* an isomorphism, because the relation $-3 + 3 = 2 \cdot 0$ in A_1 does not yield a relation for preimages in A : $(0, 3) + (1, 3) \neq 2 \cdot (0, 0)$. In the present case we have

$$T(A_1) = 31, \quad V(A) = 11, \quad V(A_1) = 25.$$

Therefore,

$$V(A) < V(A_1), \quad \text{but} \quad T(A_1) < T(A).$$

We have $\dim A = 2$. Since when we pass from A to $A_1 = \phi(A)$ we add one new relation (2.7), we have (see 2.1)

$$\dim A_1 = 1.$$

Thus, the example we just considered illustrates our theorem.

2.4. Let A be depicted with respect to an orthonormal system of coordinates. Let $\bar{a} = (a_1, a_2)$ and $\bar{b} = (b_1, b_2)$ be two points in A , which lie at the maximal distance (among the points of A) from one another (see Figure 3).

Let us draw a straight line L_1 through the point \bar{a} and a straight line L_2 parallel to it through the point \bar{b} . We choose the lines L_1 and L_2 so that A lies in the strip bounded by L_1 and L_2 , and there is a point \bar{c} in A which lies on L_1 or on L_2 . Mapping the system of coordinates with the axes L_1 and L_2 into the orthonormal system of coordinates yields a map of the set A onto a set B which satisfies condition 1) in Section 2.4.

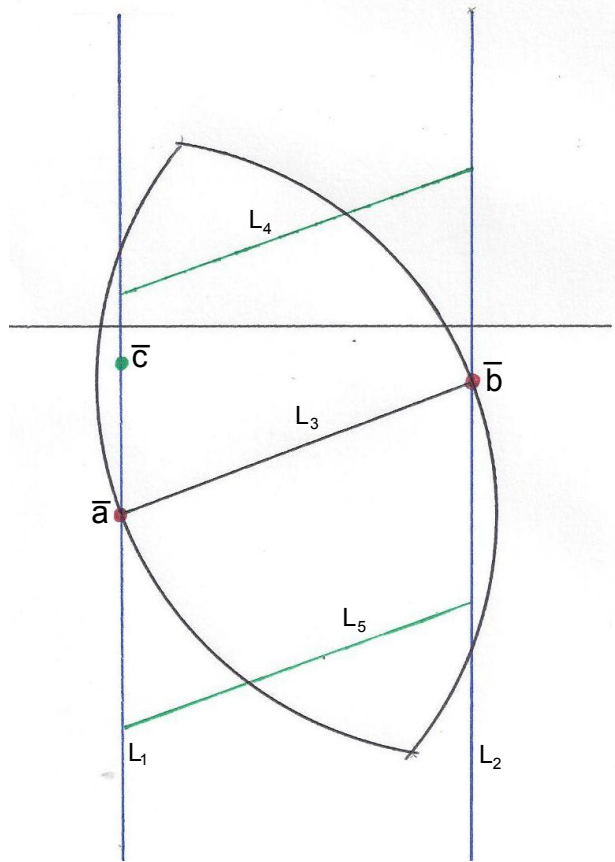


Figure 3

If we extend the example in Section 2.3 to a general case, the set A lies in a strip bounded by the lines $y = 0$ and $y = h_2 - 1$. If A is such that

1) the points $(0, 0)$ and $(0, h_2 - 1)$ belong to A ,

and

2) the point $(1, h_2 - 1)$ belong to A ,

then passing from A to $A_1 = \phi(A)$ is done in the same manner as in the numerical example.

Conditions 1) and 2) in Section 2.4 cover only part of the general case presented in Section 2.7. However, here we get stronger results concerning the quantity $V(A_1)/V(A)$ and the way the presentation leads us to the core of the problem is considerably simpler than in the case treated in Section 2.7.

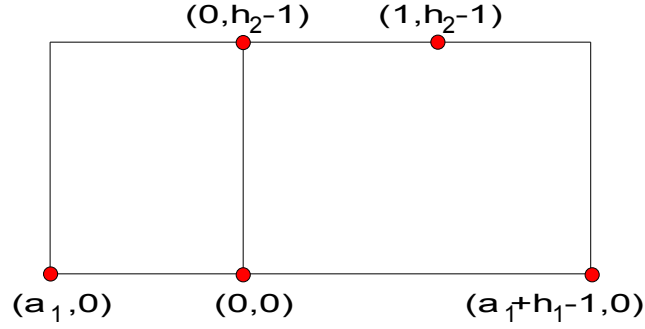


Figure 4

The cases when 1) and one of the conditions $(-1, h_2 - 1) \in A$, $(1, 0) \in A$, or $(-1, 0) \in A$ are satisfied are reduced to the case already considered by means of the reflection with respect to the line $x = 0$ or to the line $y = (h_2 - 1)/2$. Needless to say, the shift of A along the line $y = 0$ does not lead to new cases.

The set A with $\dim A = 2$ is placed in the parallelepiped $D(A)$ with minimal volume $V(D)$. According to Definition 2,

$$D = (a_1, a_2) + \{(x_1, x_2) \in \mathbb{Z}^2, 0 \leq x_1 < h_1, 0 \leq x_2 < h_2\}. \quad (2.8)$$

To satisfy the conditions 1) and 2) we put in (2.8) $a_2 = 0$ and take a_1 such that $a_1 \leq 0$ and $a_1 + h_1 - 1 \geq 0$.

Let us project the set A onto the line $x = 0$ parallel to the vector $(1, 2(h_2 - 1))$. The matrix of this map is

$$M = \begin{pmatrix} 0 & -2(h_2 - 1) \\ 0 & 1 \end{pmatrix}$$

so that

$$(x, y) \begin{pmatrix} 0 & -2(h_2 - 1) \\ 0 & 1 \end{pmatrix} = (0, y - 2(h_2 - 1)x).$$

On the line $x = a_1$ there are points of the set A . Let (a_1, y_1) be the point with the maximal value of y among those points.

In the same way we see that on the line $x = a_1 + h_1 - 1$ there are points of A . Let $(a_1 + h_1 - 1, y_0)$ be the point with the minimal value of y among those points.

The images of these two points under the projection are

$$(a_1, y_1) \begin{pmatrix} 0 & -2(h_2 - 1) \\ 0 & 1 \end{pmatrix} = (0, y_1 - 2(h_2 - 1)a_1)$$

and

$$(a_1 + h_1 - 1, y_0) \begin{pmatrix} 0 & -2(h_2 - 1) \\ 0 & 1 \end{pmatrix} = (0, y_0 - 2(h_2 - 1)(a_1 + h_1 - 1)),$$

respectively.

The projected set $A_1 = A_{\text{proj}}$ on the line $x = 0$ is such that the ordinates y of its points satisfy

$$y_0 - 2(h_2 - 1)(a_1 + h_1 - 1) \leq y \leq y_1 - 2(h_2 - 1)a_1.$$

The length of this segment is equal to

$$y_1 - 2(h_2 - 1)a_1 - y_0 + 2(h_2 - 1)(a_1 + h_1 - 1) + 1 = y_1 - y_0 + 1 + 2(h_2 - 1)(h_1 - 1).$$

The segment is minimal when $y_1 = 0$ and $y_0 = h_2 - 1$, and then the minimal length is

$$2(h_2 - 1)(h_1 - 1) - (h_2 - 1) + 1 = (2h_1 - 1)(h_2 - 1) + 1.$$

There is no arithmetic progression with difference $d \geq 2$ that contains the set A_1 ; indeed, otherwise $D(A)$ would not be minimal.

We conclude that

$$V(A_1) \geq (2h_1 - 1)(h_2 - 1) + 1 > h_1 h_2 = V(A).$$

As in Example 1, we see immediately that

$$T(A) > T(A_1)$$

and

$$\dim A_1 = 1.$$

In our case the value $V(A_1)$ is maximal compared with other possible cases.

2.5. Suppose that condition 1) of Section 2.4 is fulfilled, and 2) $(1, y_0) \in A$, where y_0 is maximal on the line $x = 1$ and

$$y_0 \geq \frac{1}{2}(h_2 - 1). \quad (*)$$

The case when $y_0 \leq \frac{1}{2}(h_2 - 1)$ can be studied by using reflection with respect to the line $y = \frac{1}{2}(h_2 - 1)$. The case when $y_0 = h_2 - 1$, was studied in Subsection 2.4, is of course a particular case of the case studied below in Subsection 2.5, but we provided it in order to illustrate the proof of Theorem 2 on a simple example, which, incidentally, also gives the bigger value of $V(A_1)$.

Let us first illustrate our general case by means of the following example.

Example 2. Consider the set (see Figure 5).

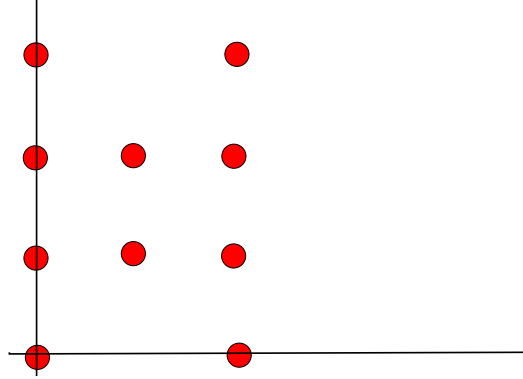


Figure 5

$$A = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (2, 3)\}.$$

In this case $k = 10$ and $T = 33$.

Let us project A onto the line $x = 0$ parallel to the vector $(1, 5)$. We obtain the set $A_1 = \phi(A)$, where $\phi(1, 2) = (0, -3)$, $\phi(1, 1) = (0, -4)$, $\phi(2, 3) = (0, -7)$, $\phi(2, 2) = (0, -8)$, $\phi(2, 1) = (0, -9)$, $\phi(2, 0) = (0, -10)$. The points on the line $x = 0$ remain on place.

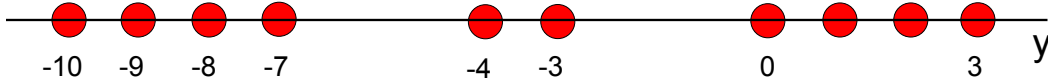


Figure 6

We have $V(A) = 12$ and $V(A_1) = 14$. Moreover, $T(A) = 33$, while $T(A_1) = 32$. Therefore,

$$V(A) < V(A_1)$$

and

$$T(A) > T(A_1).$$

2.6. Now let $\ell = (1, y_1)$, where $y_1 = h_2 - 1 + y_0$.

Let us project the set A onto the line $x = 0$ parallel to the vector ℓ . Let y_0 be the maximal number characterizing the points in conditions 1) and 2) of Subsection 2.5 (this condition may be satisfied by using a suitable isomorphic transformation; it give the set A_1 with maximal volume $V(A_1)$).

The projection in question is defined by the matrix

$$M = \begin{pmatrix} 0 & -(h_2 - 1 + y_0) \\ 0 & 1 \end{pmatrix},$$

so that

$$\phi(x, y) = (x, y) \begin{pmatrix} 0 & -(h_2 - 1 + y_0) \\ 0 & 1 \end{pmatrix} = (0, y - (h_2 - 1 + y_0)x).$$

Let

$$I_a = \{(x, y) | x = a, 0 \leq y \leq h_2 - 1\}.$$

Then

$$\phi(I_a) = \{(x, y) | x = 0, -(h_2 - 1 + y_0)a \leq y \leq h_2 - 1 - (h_2 - 1 + y_0)a\}.$$

Thus we can write

$$\phi(D) = \bigcup_a \phi(I_a).$$

Now let us consider, in the same way as in Subsection 2.4, the points (a_1, y_1) and $(a_1 + h_1 - 1, y_2)$. The images of these two points under ϕ are

$$(a_1, y_1)M = (0, y - a_1(h_2 - 1 + y_0))$$

and

$$(a_1 + h_1 - 1, y_2)M = (0, y_2 - (a_1 + h_1 - 1)(h_2 - 1 + y_0)),$$

respectively.

The projected set A_1 on the line $x = 0$ is such that for $y \in A_1$ we have

$$y_1 - a_1(h_2 - 1 + y_0) \leq y \leq y_2 - (a_1 + h_1 - 1)(h_2 - 1 + y_0).$$

The length of this segment is

$$y_2 - (a_1 + h_1 - 1)(h_2 - 1 + y_0) - y_1 + a_1(h_2 - 1 + y_0) + 1 = y_2 - y_1 + (h_2 - 1 + y_0)(h_1 - 1).$$

This segment has a minimal length when $y_1 = 0$ and $y_2 = h_2 - 1$, and then its length is

$$(h_2 - 1 + y_0)(h_1 - 1) - (h_2 - 1) + 1.$$

We conclude that it holds that

$$V(A_1) \geq (h_2 - 1 + y_0)(h_1 - 1) - (h_2 - 1) + 1 > h_1 h_2 = V(A). \quad (**)$$

In view of (*), we have

$$\begin{aligned} (h_2 - 1 + y_0)(h_1 - 1) - (h_2 - 1) + 1 &\geq (h_2 - 1)(h_1 - 2) + \frac{h_2 - 1}{2}(h_1 - 1) + 1 = \\ &= h_1 h_2 - h_1 - 2h_2 + 3 + \frac{h_2 - 1}{2}(h_1 - 1). \end{aligned}$$

To get (**) we simply verify that

$$\frac{(h_2 - 1)(h_1 - 1)}{2} - h_1 - 2h_2 - 3 > 0,$$

or

$$\frac{(h_2 - 3)(h_1 - 1)}{2} - 2h_2 + 2 > 0,$$

or, further

$$(h_2 - 3)(h_1 - 1) - 4(h_2 - 3) - 8 > 0$$

or

$$(h - 3)(h_1 - 5) > 8.$$

If $h_2 > 3$ and $h_1 > 13$ this last inequality is satisfied, and the last conditions are fulfilled when k is sufficiently large and $h_2 > 3$. The analysis of the case $h_2 = 3$ is left as an exercise.

2.7. Suppose now that condition 1) of Section 2.4 is fulfilled and

$$2) \quad (x_0, y_0) \in A,$$

where $|x_0|$ is minimal among the points $(x, y) \in A$, for which $x \neq 0$ and

$$|x_0| > 1.$$

We can assume (using reflexion) that

$$x_0 > 0.$$

Let us take all the $\{x\}$ -coordinates of points of A such that

$$a_1 \leq x \leq a_1 + h_1 - 1$$

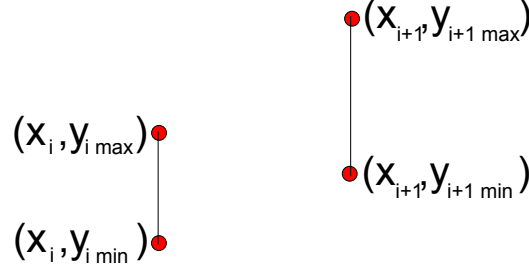
and $a_1 \leq 0$, and label them as

$$x_1 = a_1, x_2, \dots, x_r = a_1 + h_1 - 1.$$

For the points $(x_i, y) \in A$, denote $\max_{(x_i, y) \in A} y = y_{i \max}$ and $\min_{(x_i, y) \in A} y = y_{i \min}$.

Consider also the points

$$(x_{i+1}, y_{i+1 \min}) \quad \text{and} \quad (x_{i+1}, y_{i+1 \max})$$



Further, let us denote

$$\Delta_i^x := x_{i+1} - x_i \tag{2.9}$$

and

$$\Delta_i^y := \max \{y_{i \max} - y_{i+1 \min}, y_{i+1 \max} - y_{i \min}\}. \tag{2.10}$$

Note that by using, if necessary, the reflection of A with respect to the line $y = (h_2 - 1)/2$, we in fact have that

$$\Delta_i^y := y_{i+1 \max} - y_{i \min}, \tag{2.11}$$

and we will assume this from now on.

Further, let us denote

$$\tau = \min_i \frac{\Delta_i^x}{(\Delta_i^y + h_2 - 1)} \tag{2.12}$$

In this section we will consider only the case when there exists an i such that

$$\Delta_i^x = 1. \tag{2.13}$$

Needless to say, there may be several values of i such that (2.13) holds. We denote by i_0 that value of i which gives the minimum in τ .

Now let us consider the projection ϕ along the vector

$$\ell := (1, \Delta_{i_0}^y + h_2 - 1).$$

onto the line $x = 0$ (in the definition of ℓ we take $y_1 = \Delta_{i_0}^y + h_2 - 1$ instead of $y_1 = h_2 - 1 + y_0$, as in Subsection 2.6).

The proof of the inequality $V(A_1) \geq V(A)$ is completed as in Subsection 2.6, under the additional condition

$$\Delta_{i_0}^y \geq 1. \quad (2.14)$$

Note that we have

$$\phi(x+1, y_{x+1 \max}) = \phi(x, y_{x+1 \max} - (h_2 - 1) - (y_{x+1 \max} - y_{x \max})) = \phi(x, y_{x \max} - (h_2 - 1)).$$

The points $(x, y_{x \max})$ and $(x, y_{x \max} - (h_2 - 1))$ give for ordinates the same difference as for the original points $(0, 0)$ and $(0, h_2 - 1)$, i.e., in A_1 we have a new relation compared with A , and

$$\dim A_1 = 1.$$

2.8.

Let us explain this connection in more detail.

Define, using the notations in Section 2.6,

$$H_a = \{(x, y) | x = 0, h_2 - 1 - (h_2 - 1 + y_0)a \leq y \leq (h_2 - 1 + y_0)(a + 1)\},$$

for

$$a_1 \leq a \leq a_1 + h_1 - 2.$$

The points in the set $A_1'' = \bigcup_{a=a_1}^{a_1+h_1-2} H_a$ have no preimages under the map ϕ , and the points of the set $A_1' = A_1 \setminus A_1''$ have such preimages, and this fact explains why $V(A) \leq V(A_1)$.

In the case considered in our Subsection 2.7 we have a corresponding set B_1'' which has no preimages, and which is defined in the following manner:

$$B_1'' = \bigcup_{a=a_1}^{a_1+h_1-2} F_a,$$

where

$$F_a = \{(x, y) | x = 0, h_2 - 1 - (h_2 - 1 + \Delta_{i_0}^y)a \leq y \leq (h_2 - 1 + \Delta_{i_0}^y)(a + 1)\}. \quad (2.15)$$

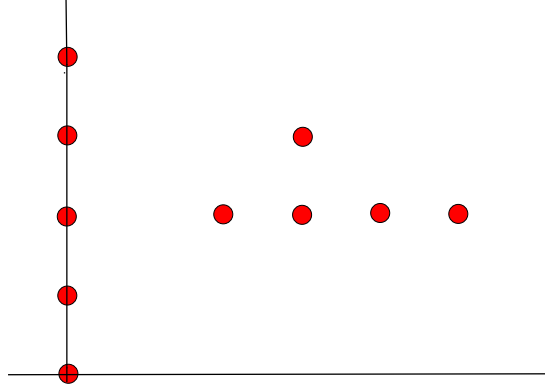
In order for the set of admissible y in (2.15) to be non-empty it is necessary that either (2.14) holds or $\Delta_{i_0}^y = 0$. In the case where (2.14) holds we have $V(A_1) \geq V(A)$.

2.9. Let us give an example for the case considered in Subsection 2.8.

Example. Consider the set

$$A = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (2, 2), (3, 2), (3, 3), (4, 2), (5, 2)\}$$

or, graphically



Taking the projection ϕ onto the line $y = 0$ along the vector $(1, 5)$ we obtain the set

$$A_1 = \phi(A) = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, -8), (0, -12), (0, -13), (0, -18), (0, -23)\}.$$

We have

$$V(A) \leq 5 \cdot 6 - 2 = 28 = V(A_1)$$

and

$$T(A) = 25, \quad T(A_1) = 24, \quad \text{i.e., } T(A_1) < T(A).$$

Thus, Theorem 2 is proved.

2.10. When we are studying the case where $\Delta_{i_0}^y = 0$, we chose the vector ℓ defining the projection without asking that the value $|x_0|$ be minimal; rather, we just keep the requirement that the projection be one-to-one.

We explain this with the help of the following example. Consider the set

$$A = \{(0, 0), (0, 1), (0, 2), (0, 4), (0, 8), (3, 8), (4, 8), (5, 8), (6, 8), (9, 8), (9, 4), (9, 2), (9, 1), (9, 0)\}$$

Here $V(A) = 90$.

Let us contract the set A along the x -axis: $\phi_1(x, y) = (x/3, y)$:

$$\begin{aligned} A_1 = \phi_1(A) = \{(0, 0), (0, 1), (0, 2), (0, 4), (0, 8), (1, 8), \\ (4/3, 8), (5/3, 8), (2, 8), (3, 8), (3, 4), (3, 2), (3, 1), (3, 0)\} \end{aligned}$$

Now let us project the set A onto the line by means of the vector $(1, 16)$. We obtain:

$$A_2 = \phi_2(A_1) = \{(0, 8), (0, 4), (0, 2), (0, 1), (0, 0), (0, -8),$$

$$(0, -13 \cdot 1/3), (0, -18 \cdot 2/3)(0, -24), (0, -40), (0, -44), (0, -46)(0, -47), (0, -48)\}.$$

Finally, let us dilate the set A_2 along the y axis: $\phi_3(x, y) = (x, 3y)$. We get $A_3 = \phi_3(A_2)$

$$= \{(x, y) | x = 0, y = 24, 12, 6, 3, 0, -24, -40, -56, -72, -120, -132, -138, -141, -144\}.$$

Therefore,

$$V(A_3) = 169$$

and

$$V(A_3) > V(A).$$

In A_3 we have

$$24 + (-24) = 2 \cdot 0 = 0,$$

but for the preimages in A we have

$$(0, 8) + (3, 8) \neq 0,$$

so that $\dim A_3 = 1$.

The transition from dimension $d + 1$, with $d \geq 2$, to dimension d , is carried out in similar manner.

3 On L_m , the exact value of the extremal volume for $\mathbb{A}(k, T)$

3.1. Length of extremal sets.

In Section 3 we will study sets for which $\dim A = 1$. Such sets lie inside segments of \mathbb{Z} . The minimal length of a family of isomorphic sets is given by a set in normal form which lies in the segment $[0, a_{k-1}]$. Let us define $\mathbb{A}(k, T)$ as the set of all families of isomorphic sets with given values of k and T . Two sets in $\mathbb{A}(k, T)$ are not necessarily isomorphic. Denote

$$a_m = \max_{A \in \mathbb{A}(k, T)} a_{k-1}.$$

A set for which $a_{k-1} = a_m$ will be called *extremal*. The volume (i.e., length) of such a set is equal to

$$V(A) = L(A) = 1 + a_m.$$

Exact formula for a_m .

In [4], p. 4 the following conjecture was formulated:

If

$$T = ck - \frac{c^2 + c - 4}{2} + b, \tag{3.1}$$

where

$$2 \leq c \leq k - 1 \quad (3.2)$$

and

$$0 \leq b \leq k - c - 1, \quad (3.3)$$

then

$$a_m = 2^{c-2}(k + 1 - c + b). \quad (3.4)$$

3.2. The $T \longleftrightarrow (c, b)$ one-to-one correspondence

We will ensure that the function $a_m = \mathcal{F}(k, T)$ is well defined if we can show that the set of values $\{T\}$ and $\{(c, b)\}$ are into a one-to-one correspondence.

For a given c , the values of T are, in view of (3.1) and (3.2), situated in the segment

$$\begin{aligned} & \left[ck - \frac{c^2 + c - 4}{2}, ck - \frac{c^2 + c - 4}{2} + k - c - 1 \right] = \\ & = \left[ck - \frac{c^2 + c - 4}{2}, (c + 1)k - \frac{(c + 1)^2 + (c + 1) - 4}{2} \right]. \end{aligned}$$

T is minimal when $c = 2$ and $b = 0$, i.e.,

$$T = 2k - 1.$$

T is maximal when $c = k - 1$ and $b = 0$, i.e.,

$$T = (k - 1)k - \frac{(k - 1)^2 + (k - 1) - 4}{2} = \frac{k^2 - k}{2} + 2.$$

3.3. Concrete example. Let us give a concrete numerical example explaining how to obtain an estimate from below for $\mathcal{F}(k, T)$ (we will conjecture that this estimate is in fact sharp).

Let us take the set

$$A = \{0, 1, 2, 4, 8, 16, 32, 48, 96, 192, 384\}.$$

Here we have $|A| = 11$ and $a_m = 384$. We will calculate T in several steps, appropriate for further generalization.

Denote

$$A_1 = \{8, 16, 32, 48\} = 8 \times \{0, 1, 2, 4, 6\},$$

$$A_2 = \{1, 2, 4\} = \{2^0, 2^1, 2^2\},$$

and

$$A_3 = \{96, 192, 384\} = 96 \times \{2^0, 2^1, 2^2\}.$$

The sums $1 + a_i$ contribute 10 numbers to T , $A \setminus \{1\}$ is left and $2 + a_i$ contributes 9 sums, the number 4 gives 8 sums, 384 gives 7 sums, 192 gives 6 sums, and 96 gives 5 sums. We get the set $8 \times \{0, 1, 2, 4, 6\}$. Here $k' = 5$, $b = 2$ (the number of holes), and $T' = 10 - 1 + 2 = 11$. Consequently,

$$T = 10 + 9 + 8 + 7 + 6 + 5 + 11 = 56.$$

3.4. General example. Now it is not difficult, using the above example, to build a generalized example A . To this end we will use numbers $k_0 = k_2 + k_3$, k_1 and $b \leq k_1 - 3$. We take the set A in the form

$$A = A_1 \cup A_2 \cup A_3,$$

where

$$A_2 = \{1, 2, 2^2, \dots, 2^{k_2-1}\}, \quad A_3 = 2^{k_2} \times A'_3.$$

The set A'_1 is defined by the two numbers k_1 and b by means of the Basic Theorem. Namely, A'_1 is a part of the segment $[0, k_1 - 1 + b]$ satisfying the condition $|2A'_1| = 2k_1 - 1 + b$. The detailed description of the set A'_1 is provided by the Basic Theorem.

The last number in the set A_1 is $p = 2^{k_2}(k_1 - 1 + b)$. Now let us describe the set A_3 :

$$A_3 = 2p \times \{1, 2, 2^2, \dots, 2^{k_3-1}\}.$$

Put $k_0 = k_2 + k_3$. Computing T as in the specific example above, we get

$$\begin{aligned} T &= k - 1 + (k - 2) + \dots + (k - k_2) + (k - k_2 - 1) + \dots + (k - k_0) + 2k_1 - 1 + b = \\ &= \frac{k - k_0 + k - 1}{2} k_0 + 2k_1 - 1 + b = \\ &= k_0 k - \frac{(k_0 + 1)k_0}{2} + 2k - 2k_0 + b - 1 = (k_0 + 2)k - \frac{(k_0 + 5)k_0}{2} + b - 1. \end{aligned}$$

Denoting $k_0 + 2 = c$, we get

$$T = ck - \frac{c^2 + c - 4}{2} + b,$$

as in (3.1).

Here

$$2 \leq c \leq k - 1, \quad 0 \leq b \leq k - c - 1.$$

3.5. Not very much is known about the exact form of the function $\mathcal{F}(T) = a_m$ (only, the conjecture below).

The Basic Theorem gives us:

If $2k - 1 \leq T \leq 3k - 4$, when

$$T = 2k - 1 + b, \quad 0 \leq b \leq k - 3,$$

then

$$a_m = k - 1 + b.$$

Strangely enough, even for

$$T = 3k - 4 + b, \quad 0 \leq b \leq k - 4,$$

in which case, according to my early conjecture, it must hold that

$$a_m = 2k - 4 + 2b$$

the proof is still missing.

In a series of papers, Renling Jin (see [8], [9], [10]) studied similar problems.

4 Approximate groups

B. Green and T. Tao have introduced, in a series of seminal papers, the notion of approximate group. For given k and T , denote by $\mathbb{AG}(k, T)$ the collection of all sets that are *approximate groups in \mathbb{Z}* (Tao, Green) and have characteristic values k and T .

In our case such sets A have a simple description: A is a finite set of integers with characteristics k and T (as everywhere in the paper) and satisfy the additional conditions

- (1) A is symmetric, i.e., if $a \in A$, then $-a \in A$;
- (2) $0 \in A$.

The set $\mathbb{AG}(k, T)$ is of course partitioned into classes with respect to the relation given by isomorphism.

As it was explained in Section 2, we can focus on the study of sets A with $\dim A = 1$. In this case the volume (or length) of a set A can be calculated as follows:

We may assume that $\text{g.c.d.}(A) = 1$. Then there exists a segment $L_m = [-b_m, b_m]$ such that $A \subset L_m$ and L_m has minimal length. Then

$$V(A) = V(L_m) = 2b_m + 1.$$

We propose the following

Conjecture. *For a set $A \in \mathbb{AG}(k, T)$ with T written in the form*

$$T = ck - \frac{3c^2 - 2c - 4}{2} + b$$

we have

$$L_m = 3^{\frac{c}{2}-1}(k - c + b + 1) + 1, \tag{4.1}$$

with

$$2 \leq c \leq k-1, \quad 0 \leq b \leq k-c-1.$$

Let us construct a set for which L is given by the right-hand side of (3.5). This example will show that L_m cannot be smaller, i.e.,

$$L_m \geq 3^{\frac{c}{2}-1}(k-c+b+1)+1.$$

Let $|A| = k$, k odd, and assume A is symmetric with respect to 0. Suppose the first \bar{k}_1 elements, $|A_0| = \bar{k}_1$, obey the Basic Theorem. This means that (b is even)

$$T_0 = 2\bar{k}_1 - 1 + b \quad L = \bar{k}_1 + b. \quad (4.2)$$

A_0 lies in the following segment:

$$A \subset \left[-\frac{\bar{k}_1 - 1 + b_1}{2}, \frac{\bar{k}_1 - 1 + b_1}{2} \right] = [p, p].$$

Let us add here two points, $-3p$ and $3p$, then another two points $-9p$ and $9p$, and so on. Overall, we add $2\bar{k}_2$ points, so we have the segment

$$\left[-3^{\bar{k}_2}p, 3^{\bar{k}_2}p \right].$$

We have a total of k points, with

$$k = \bar{k}_1 + 2\bar{k}_2.$$

The set A lies in a segmeny of length L (i.e., the segment contains L points), where

$$\begin{aligned} L &= 3^{\bar{k}_2} \cdot 2p + 1 = 3^{\bar{k}_2} \cdot 2 \frac{\bar{k}_1 - 1 + b_1}{2} + 1 = \\ &= 3^{\bar{k}_2} (\bar{k}_1 + b - 1) + 1 = 3^{\bar{k}_2} (k - 2\bar{k}_2 + b - 1) + 1. \end{aligned}$$

Denoting $c = 2\bar{k}_2 + 2$, we have

$$L = 3^{\frac{c}{2}-1}(k-c+b+1)+1.$$

From the Basic Theorem it follows that

$$0 \leq b \leq \bar{k}_1 - 3.$$

Now let us calculate T . For the first \bar{k}_1 points we assume that the Basic Theorem holds true, so that

$$T_0 = 2\bar{k}_1 - 1 + b, \quad 0 \leq b \leq \bar{k}_1 - 3$$

(b is the number of holes, $b/2$ holes to the left, and $b/2$ to the right). Let us add two more points (see (3.6)); then to T_0 one adds $2\bar{k}_1$. Indeed, after we add the pair of points

$$-3p, \dots, -p, 0, \dots, p, 3p,$$

we have $3p + (-p) = 2p$, $3p + (-3p) = 0$. From $-p$ to p there are \bar{k}_1 points: $3p$ with these points, and $-3p$ of these points contribute $2\bar{k}_1$ to T_0 . Adding two more points yields

$$2\bar{k}_1 + 2 = 2(\bar{k}_1 + 1).$$

Further, we get

$$2\bar{k}_1 = 2(\bar{k}_1 + 2).$$

Let us a pair \bar{k}_2 times; the additions to T are given by

$$2\bar{k}_1, 2(\bar{k}_1 + 1), 2(\bar{k}_1 + 2), \dots, 2(\bar{k}_1 + \bar{k}_2 - 1).$$

The last pair was k .

Altogether we get

$$\begin{aligned} T &= T_0 + 2\bar{k}_1 + 2(\bar{k}_1 + 1) + \dots + 2(\bar{k}_1 + \bar{k}_2 - 1) = \\ &= T_0 + 2\bar{k}_2 \frac{\bar{k}_1 + \bar{k}_1 + \bar{k}_2 - 1}{2} = \end{aligned}$$

Here there are \bar{k}_2 terms (in an arithmetic progression). The first term is $2\bar{k}_1$, and the last is $2(\bar{k}_1 + \bar{k}_2 - 1)$, so that continuing we have

$$= T_0 + \bar{k}_2(2\bar{k}_1 + \bar{k}_2 - 1) = T_0 + \bar{k}_2(2k - 3\bar{k}_2 - 1) = 2\bar{k}_1 = 1 + b + \dots.$$

We have

$$\bar{k}_2 = \frac{2\bar{k}_2 + 2}{2} - 1 = \frac{c}{2} - 1,$$

so that

$$\begin{aligned} T &= 2k - 4\bar{k}_2 - 1 + b + \left(\frac{c}{2} - 1\right) \left(2k - \frac{3c}{2} + 3 - 1\right) = \\ &= 2k - 2c + 4 - 1 + b + ck - 2k - \frac{3c^2}{4} + \frac{3c}{2} + c - 2 = \\ &= ck + \frac{c}{2} - \frac{3c^2}{4} + 1 + b = ck - \frac{3c^2 - 2c - 4}{4} + b. \end{aligned}$$

This is in agreement with the above conjecture.

5

In conclusion, let us note that the description of the exact structure of extremal sets in the Abelian case will clearly lead to similar progress in the non-Abelian case. The papers [9], [10], and [11] give some hints on how to begin such a study.

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